Scattering from Three-Dimensional Cavity-Backed Apertures in an Infinite Ground Plane by RBCI

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Abstract—This paper describes an extension of the Robin Boundary Condition Iteration (RBCI) method to solve the problem of scattering from three-dimensional cavities embedded in an infinite ground plane. The proposed method is a hybrid one since it combines a differential equation for the interior and the neighborhood of the cavities with an integral equation for the rest of the unbounded domain. A suitable choice of the boundary condition (of the nonhomogeneous mixed type) on the fictitious boundary dividing the two parts of the domain avoids resonances whatever the frequency. Moreover, an iterative solver is described for the efficient solution of the discretized global system of linear equations.

Index Terms—Electromagnetic scattering, three-dimensional cavity-backed apertures, finite element methods.

I. INTRODUCTION

THE computation of time-harmonic electromagnetic scattering from three-dimensional cavity-backed apertures in an infinite ground plane has attracted the attention of various researchers [1], [2] and is also treated in some textbooks on computational electromagnetics [3].

When the cavity is such that its Green’s function is not known, almost all the methods proposed in literature belong to the hybrid FEM/BEM (Finite Element Method/Boundary Element Method) family. To the authors’ knowledge, only in [4] is an ABC (Absorbing Boundary Condition)-like method used to deal with microstrip antennas on a conducting platform. Although the ABC method seems to be applicable to scattering from cavity-backed apertures in an infinite ground plane, FEM/BEM is still preferred.

When the FEM/BEM method is applied to the problem of scattering from cavity-backed apertures in an infinite ground plane, the cavity is terminated at its aperture where a magnetic current distribution is assumed as a new unknown for the problem [3]. The interior of the cavity is treated by conventional FEM, whereas the exterior is treated by means of an integral equation which takes into account the magnetic current distribution mentioned above. The two sets of equations are then coupled and solved. Note that the termination of the cavity at its aperture is not strictly necessary; in fact the terminating surface can be chosen differently at the cost of a more involved formulation.

Recently the authors devised a method, called the Robin Boundary Condition Iteration (RBCI) procedure [5], [6] for the solution of 2-D electromagnetic scattering problems, which is based on the introduction of a fictitious boundary, enclosing the scatterer, on which a nonhomogeneous Robin (mixed) boundary condition is first guessed and then iteratively improved by means of Green’s formula until convergence takes place. The RBCI method has also been extended to 3-D electromagnetic scattering problems [7] by employing tetrahedral edge elements. A similar approach has been successfully applied to the solution of electrostatic [8], [9] and time-harmonic skin effect [10] problems in open boundaries, in which a Dirichlet boundary condition is used on the fictitious boundary.

In this paper RBCI is adapted to scattering from three-dimensional cavity-backed apertures in an infinite ground plane.

The paper is organized as follows. In Section II the proposed hybrid FEM/RBCI method is described. In Section III a validation example is given. The authors’ conclusions follow in Section IV.

II. THE RBCI METHOD FOR ELECTROMAGNETIC SCATTERING FROM CAVITY-BACKED APERTURES

Consider a system of one or more cavity-backed apertures in a Perfectly Electrical Conducting (PEC) infinite plane, lit up by an incident monochromatic wave $\phi_0$ like the one depicted in Fig. 1. An unbounded scattering problem is set up in terms of

Fig. 1. A cavity on a ground plane.
the total field $\phi$, which satisfies the vector Helmholtz equation [11]
\[
\nabla \times (\varepsilon_r^{1/2} \nabla \times \phi) - j k_0 \sigma \phi = 0
\]
where $\phi$ may represent the electric or the magnetic field. For the former case $\epsilon_r$ and $\mu_r$ represent the relative magnetic permeability $\mu_r$, and the relative electric permittivity $\epsilon_r$, whereas for the latter case their meaning is interchanged. In both cases $k_0$ is the free space wavenumber $k_0 = \omega / (\varepsilon_0 \mu_0)^{1/2}$, where $\omega$ is the wave angular frequency and $\varepsilon_0$ and $\mu_0$ are the free-space electrical permittivity and magnetic permeability, respectively.

A homogeneous Dirichlet condition ($\mathbf{n} \times \phi = 0$) holds on the PEC surfaces $\Gamma_C$ when $\phi$ represents the electric field, whereas a homogeneous Neumann condition ($\mathbf{n} \cdot \nabla \phi = 0$) holds on $\Gamma_C$ when $\phi$ represents the magnetic field. Moreover, the scattered field $\phi_S$ must satisfy the Sommerfeld radiation condition [11].

With the aim of computing the field near and inside the cavities, let us introduce a fictitious boundary $\Gamma_F$ having the border lying entirely on the PEC plane and enclosing the cavity’s apertures (see Fig. 1). On it a nonhomogeneous Robin (mixed) boundary condition is assumed:
\[
\mathbf{n} \times \phi + j k_0 \mathbf{n} \times (\mathbf{n} \times \phi) = \psi \quad \text{on} \quad \Gamma_F
\]
where $\mathbf{n}$ is the outward normal to $\Gamma_F$ and $\psi$ is an unknown vector function on $\Gamma_F$.

By discretizing the domain $D$, delimited by $\Gamma_F$ and $\Gamma_C$ using tetrahedral edge elements, a linear algebraic system is built:
\[
A \Phi = C \Psi
\]
where $A$ is a square sparse symmetric matrix, $\Phi$ is the vector of the unknown field values (including those of the edges lying on $\Gamma_F$), $C$ is a rectangular matrix in which null columns appear for the edges not belonging to the elements external to $\Gamma_M$ and having a side lying on it. Note that $\Gamma_M$ can be selected as coinciding with the cavity’s apertures $\Gamma_A$: in this case the first integral on the right-hand side of (4) vanishes if $\phi$ represents the electric field, whereas the second integral on the right-hand side of (4) vanishes if $\phi$ represents the magnetic field.

Taking the curl of (4) and using some elementary manipulation, a formula similar to (4) can be obtained for $\nabla \times \phi$ [11], so that function $\psi$ in the Robin condition (2) can be expressed from the field inside $\Gamma_F$ as:
\[
\psi(\mathbf{r}) = \Re(\phi_0(\mathbf{r}) + \phi_r(\mathbf{r}')) + \int_{\Gamma_M} \Re \mathbf{G} \cdot (\mathbf{n}' \times \nabla' \times \phi(\mathbf{r}'))
\]
\[
+ \nabla' \times \Re \mathbf{G} \cdot (\mathbf{n}' \times \phi(\mathbf{r}')) \, ds'
\]
\[
(8)
\]
In the FEM approximation this formula reads as:
\[
\Psi = \Psi_0 + M \Phi
\]
\[
(9)
\]
where $\Psi_0$ results from he he discretization of $\Re \{\phi_0 + j \phi_r\}$ and $M$ is a rectangular matrix in which null columns appear for the edges not belonging to the elements external to $\Gamma_M$ and having a side lying on it.

The global system of linear algebraic Eqs. (3) and (9) can be conveniently solved by iteration: initially guessing $\Psi$ (a good choice is $\Psi_0$), Eq. (3) is solved for $\Phi$; Eq. (9) is then used to improve $\Psi$. This procedure is repeated until convergence takes place, i.e. when the ratio of the norm of the difference of two consecutive solutions with the norm of the current solution is smaller than a user-defined tolerance $\delta$ [6].

As pointed out in [6], this formulation completely avoids resonances: in fact (1) cannot have nontrivial solutions satisfying the homogeneous boundary condition, because the Robin (i.e. impedance) condition works as an energy absorber. For the same reason $\Re \{\phi_0 + j \phi_r\}$ vanishes on the fictitious boundary $\Gamma_F$ if and only if $\phi_0 + j \phi_r$ vanishes inside $\Gamma_F$, so that resonances are avoided in the integral equation as well.

Note that it is possible to use integration surfaces $\Gamma_M$ made up of several separated parts and also fictitious boundaries $\Gamma_F$ made up of separated parts when more than one cavity backed aperture is present in the problem at hand. This choice reduces the amount of memory requirements and computing time, since it maximizes the air to be meshed outside the cavities. Moreover, the procedure can be made more efficient if the following points are fully exploited in implementation.

i) Since the FE mesh remains unchanged through the various iteration steps, matrices $A$, $M$ and $C$ do not change: they are computed only once, at the beginning of the procedure and saved for further use.

ii) Equation (3) may be solved efficiently by means of standard solvers, which exploit the matrix $A$ sparsity and symmetry. It is to be noted that the iterative solver used to solve Eq. (3) starts with a good initial guess for $\Phi$, coming from the last RBCI iteration, because as the algorithm converges $\Phi$ does not change too much.

iii) The end-iteration test is conveniently restricted to the fictitious boundary, thus assuring convergence of the field solution in the domain.
iv) A good initial guess for $\Psi$ is $\Psi_0$ since the number of iterations is minimized (note that the Robin operator $R$ selected is like an ABC one).

By implementing the above items the proposed solution scheme is very competitive with respect to other techniques as far as computing time and memory requirements are concerned [6]: only a few iterations (generally 6–8 for a termination tolerance $\delta$ equal to 1%) are needed to reach convergence with a very small gap between $\Gamma_M$ and $\Gamma_F$ ($\lambda/20$ is generally sufficient).

In addition the procedure is easily implementable in a pre-existing FEM code for bounded problems. Only one routine has to be developed which calculates the matrix $M$ entries (very often a similar routine is already available in the post-processing program). Note that no singularities arise in these calculations since $\Gamma_F$ and $\Gamma_M$ do not have points in common.

### III. A Validation Example

In this Section the proposed hybrid method will be validated by means of an example taken from [3]. It concerns a parallelepipedal empty cavity on an infinite ground plane (coinciding with the $x$-$y$ plane) lit up by a plane wave. The cavity walls are aligned with the coordinate planes and their dimensions are $0.7\lambda$, $0.1\lambda$, and $1.73\lambda$ along the $x$-, $y$- and $z$-axis, respectively.

The cavity interior is discretized with a regular mesh made up of 4900 tetrahedral edge elements of the first order.

The integration surface $\Gamma_M$ is selected as coinciding with the cavity’s aperture and is regularly discretized with 56 triangular patches. The fictitious boundary $\Gamma_F$ is chosen as a brick surface over the PEC plane with dimensions $0.9\lambda$, $0.3\lambda$, and $0.1\lambda$, i.e. it is put at a mean distance of $0.1\lambda$ from the integration surface $\lambda_M$. Two layers of elements are used to discretize the gap enclosed by $\Gamma_M$, $\Gamma_F$ and the PEC plane, so that a further 1080 tetrahedral elements are added to the previous mesh. The total number of edges, including those with assigned value, is 9385 (see Fig. 2).

Introducing a spherical coordinate frame $(r, \theta, \phi)$ the incident electric field is assigned as:

$$\phi_0(r) = (\theta \cos \alpha + \varphi \sin \alpha) e^{-jkx}$$

where $\alpha$ is the polarization angle and $k$ is the propagation vector, given by:

$$k = -j_0(x \sin \theta_0 \cos \varphi_0 + y \sin \theta_0 \sin \varphi_0 + z \cos \theta_0)$$

where $\theta_0$ and $\varphi_0$ are the incidence angles.

The co-polarized and cross-polarized backscattered radar cross sections (RCS) were computed for $\theta_0 = 40^\circ$ and $\varphi_0$ ranging from $0^\circ$ to $90^\circ$, with $\alpha = 0^\circ$ ($E_0$ polarization) and $\alpha = 90^\circ$ ($E_c$ polarization).

For each incident field a separate run was executed. The COCG (Complex Conjugate Gradient) [12] solver with diagonal preconditioning was used as the inner solver, i.e. to repeatedly solve Eq. (3).

The end-iteration tolerances were set equal to 0.01% for the RBCI procedure and 0.005% for the COCG solver. With these parameters the mean number of RBCI iterations was equal to 11 and the mean CPU time was 2’35’’ on a 433 MHz Intel Celeron PC.

Fig. 3 shows the co-polarized and cross-polarized backscattered RCS for the $E_0$-polarized case, whereas Fig. 4 shows the co-polarized and cross-polarized backscattered RCS for the $E_c$-polarized case. A very good agreement can be observed with the results in [3], which were obtained by means of FEM/BEM.
The Robin Boundary Condition Iteration (RBCI) method is an effective hybrid method to deal with electromagnetic scattering problems by means of the finite element method.

In this paper we have adapted it to the case of scattering from cavity-backed apertures in an infinite ground plane. Due to the great generality of the method and its simplicity of implementation, the adaptation is straightforward.

The method avoids internal resonances without losing the symmetry of the FEM part of the system, and efficiently solves the global system of linear algebraic equations in an iterative way. The error in the solution is only due to the finite element discretization and to the convergence tolerance, which are selected by the user. Another advantage of the procedure lies in the simplicity of implementation in a pre-existing FEM code for bounded problems.

The RBCI procedure has been implemented in ELFIN, a large FEM code developed by the authors for electro-magnetic CAD research [13].

REFERENCES